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On the non-linear Lamb–Taylor instability

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A non-linear analysis of the inviscid stability of the common surface of two superposed fluids is presented. One of the fluids is a liquid layer with finite thickness having one surface adjacent to a solid boundary whereas the second surface is in contact with a semi-infinite gas of negligible density. The system is accelerated by a force normal to the interface and directed from the liquid to the gas. A second-order expansion is obtained using the method of multiple time scales. It is found that standing as well as travelling disturbances with wave-numbers greater than

$$k_c' = k_c[1+\tfrac{3}{8}a^2k_c^2+\tfrac{51}{512}a^4k_c^4]^{\frac{1}{2}},$$

where a is the disturbance amplitude and k_c is the linear cut-off wave-number, oscillate and are stable. However, the frequency in the case of standing waves and the wave velocity in the case of travelling waves are amplitude dependent. Below this cut-off wave-number disturbances grow in amplitude. The cut-off wave-number is independent of the layer thickness although decreasing the layer thickness decreases the growth rate. Although standing waves can be obtained by the superposition of travelling waves in the linear case, this is not true in the non-linear case because the amplitude dependences of the wave speed and frequency are different. A mechanism is proposed to explain the over-stability behaviour observed by Emmons, Chang & Watson (1960).

1. Introduction

Lamb (1932, § 231) pointed out the unstable behaviour of the common boundary of two semi-infinite inviscid incompressible fluids in a normal acceleration field directed from the denser to the lighter fluid. His results indicate that, in the absence of surface tension and viscosity, an oscillatory initial disturbance grows exponentially with time. The rate of growth is given by

$$[g'k'(\rho-\rho')/(\rho+\rho')]^{\frac{1}{2}},$$

where g' is the acceleration, k' is the wave-number of the disturbance, and ρ and ρ' are the densities of the denser and lighter fluids respectively. The results of Lamb (1932, § 267) indicate that surface tension stabilizes disturbances with wave-numbers above $k_c = [g'(\rho-\rho')/T]^{\frac{1}{2}}$, where T is the surface tension.

Taylor (1950), neglecting surface tension, extended the analysis of Lamb to the case where the denser fluid has a finite thickness with incompressible fluids on both sides. Although one of the surfaces of the heavier fluid is stable, it was assumed initially to be flat but was allowed to distort. Lewis (1950) reported experimental results on the instability of the interface of two fluids conducted with large accelerations normal to the interface so that surface tension effects can be neglected. These results agree with the analysis of Taylor within experimental scatter.

Bellman & Pennington (1954) showed that the effect of surface tension is to produce a cut-off wave-number above which disturbances oscillate and below which disturbances grow. They also found that viscosity dampens the oscillatory disturbances, and diminishes the rate of growth of unstable disturbances. However, viscosity by itself cannot make the rate of growth go to zero. Allred, Blount & Miller (1954) performed experiments, using two fluids with densities close to each other, to study the effects of surface tension. Their growth rates are a factor of two below those of Bellman & Pennington.

All of these theories are applicable for short times (to amplitudes of the order of $0{\cdot}4\lambda$, where λ is the disturbance wavelength). Beyond this amplitude, Lewis (1950) observed that the denser fluid will form narrow thin spikes into the lighter fluid, while the lighter fluid forms bubbles that move into the denser fluid. To explain these later developments, Ingraham (1954), neglecting surface tension, obtained a second-order solution which shows the distortion of sinusoidal waves.

Emmons *et al.* (1960) reported a combined experimental and analytical study. Their experimental results indicate that disturbances grow at the cut-off wave-number, k_c. Moreover, disturbances corresponding to wave-numbers above k_c exhibit an 'overstability' behaviour (a standing oscillation of the waves with amplitude increasing in time). However, only a single oscillation was observed before monotonic growth occurred. In their analysis, they neglected the lighter fluid and considered a semi-infinite fluid to obtain a third-order expansion. Their second-order term tends to infinity as the wave-number approaches k_c. Moreover, their expansion for wave-numbers larger than k_c is valid only for short times, and hence cannot be used to explain the overstability behaviour.

Rajappa (1967) presented an expansion using the method of straining of coordinates (Lighthill 1949) to obtain a uniformly valid expansion for all times. However, his analysis is not valid near k_c. Because of his invalid expansion near k_c, he obtained an incorrect amplitude dependence for the cut-off wave-number.

All of the non-linear analyses were carried out for standing waves in a semi-infinite fluid. In this paper we consider non-linear analysis of standing as well as travelling waves in a finite liquid layer. We obtain a uniformly valid expansion for large times using the method of multiple time scales (Nayfeh 1965, 1968) for wave-numbers near and larger than k_c. We determine a fourth-order cut-off wave-number. For wave-numbers less than k_c we present an expansion valid for short times. Possible mechanisms for the explanation of the overstability behaviour are discussed.

2. Mathematical formulation

We consider the stability of the interface of an inviscid liquid layer of thickness h', and a semi-infinite gas. We assume that the second face of the liquid layer is always adjacent to a solid face, and the density of the gas is small compared with that of the liquid. We assume that the flow is two-dimensional, and the x- and y-axes are in and normal to the undisturbed interface. We assume that the motion of the whole system is started from rest, and we consider a simple standing or travelling sinusoidal disturbance with amplitude a and wave-number k'. We non-dimensionalize distances, velocities, and time by $1/k'$, $(g'/k')^{\frac{1}{2}}$, and $(g'k')^{-\frac{1}{2}}$ respectively, where g' is the acceleration normal to the interface and directed from the liquid to the gas. Thus, if primes and unprimes denote dimensional and non-dimensional quantities respectively, then

$$h = h'k', \quad x = k'x', \quad y = k'y', \quad t = (g'k')^{\frac{1}{2}}t', \quad \phi = k'^{\frac{3}{2}}g'^{-\frac{1}{2}}\phi', \tag{2.1}$$

where ϕ is the velocity potential function.

In terms of these non-dimensional quantities, the velocity potential is given by

$$\phi_{xxx} + \phi_{yy} = 0, \tag{2.2}$$

for $$-\infty < x < \infty, \quad \eta > y \geqslant -h, \quad t \geqslant 0,$$

where $\eta(x,t)$ is the non-dimensional surface displacement in the y direction. At the solid/liquid interface the normal velocity vanishes, i.e.

$$\phi_y(x,y,t) = 0 \quad \text{at} \quad y = -h. \tag{2.3}$$

The liquid/gas interface moves with the liquid,

$$\eta_t - \eta_x\phi_x + \phi_y = 0 \quad \text{on} \quad y = \eta(x,t). \tag{2.4}$$

If the gas pressure is neglected, then the dynamic boundary condition at the interface is

$$-\eta - \phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) = k^2\eta_{xx}(1+\eta_x^2)^{-\frac{3}{2}} \tag{2.5}$$

on $y = \eta(x,t)$, where $$k = k'/k_c, \tag{2.6}$$

and k_c is the linearized cut-off wave-number, i.e.

$$k_c = (\rho g'/T)^{\frac{1}{2}}. \tag{2.7}$$

The parameter k can be interpreted as the ratio of the surface tension force to the gravity force. The initial conditions are

$$\eta(x,0) = \epsilon\cos x, \tag{2.8}$$

$$\eta_t(x,0) = 0, \tag{2.9}$$

where $$\epsilon = ak'. \tag{2.10}$$

Since the problem is non-linear, one cannot Fourier-analyze an arbitrary initial condition. Consequently, the present theory is applicable only to the case where the initial condition is given by (2.8).

To find an approximate solution for small ϵ to (2.2)–(2.9) we Fourier-analyze $\eta(x,t)$ and $\phi(x,y,t)$. Thus, we let

$$\begin{aligned} \eta(x,t) = \epsilon[\eta_1(t)\,e^{ix} + \bar{\eta}_1(t)\,e^{-ix}] \\ + \epsilon^2[\eta_2(t)\,e^{2ix} + \bar{\eta}_2(t)\,e^{-2ix}] \\ + \epsilon^2\tilde{\eta}_2(t) + \ldots, \end{aligned} \tag{2.11}$$

where a bar denotes complex conjugation. Hence, (2.2) and (2.3) lead to

$$\phi(x,y,t) = \epsilon[\phi_1(t)\,e^{ix} + \bar{\phi}_1(t)\,e^{-ix}]\cosh(y+h) + \epsilon^2[\phi_2(t)\,e^{2ix} + \bar{\phi}_2(t)\,e^{-2ix}]\cosh 2(y+h) + \epsilon^2\tilde{\phi}_2(t) + \dots. \tag{2.12}$$

Substituting (2.11) and (2.12) in (2.4) and (2.5) and keeping only terms of third order in ϵ lead to

$$\frac{d\eta_1}{dt} + \phi_1 \sinh h = \epsilon^2 f + O(\epsilon^3), \tag{2.13a}$$

$$f = \bar{\phi}_1\eta_2\cosh h + \tfrac{1}{2}\bar{\phi}_1\eta_1^2\sinh h - 2\phi_2\bar{\eta}_1\cosh 2h - \phi_1\eta_1\bar{\eta}_1\sinh h, \tag{2.13b}$$

$$\frac{d\phi_1}{dt}\cosh h - (k^2-1)\,\eta_1 = \epsilon^2 g + O(\epsilon^3), \tag{2.14a}$$

$$g = 2\bar{\phi}_1\phi_2\cosh 3h + 2\bar{\phi}_1\phi_1\eta_1\sinh 2h - \tfrac{3}{2}k^2\eta_1^2\bar{\eta}_1 - 2\bar{\eta}_1\phi_2'\sinh 2h - \phi_1'\eta_1\eta_1'\cosh h - \tfrac{1}{2}\eta_1^2\bar{\phi}_1'\cosh h - \bar{\phi}_1'\eta_2\sinh h, \tag{2.14b}$$

$$\frac{d\eta_2}{dt} + 2\phi_2\sinh 2h = -2\eta_1\phi_1\cosh h + O(\epsilon), \tag{2.15}$$

$$\frac{d\phi_2}{dt}\cosh 2h - (4k^2-1)\,\eta_2 = -\tfrac{1}{2}\phi_1^2 - \eta_1\phi_1'\sinh h + O(\epsilon), \tag{2.16}$$

$$\frac{d\tilde{\eta}_2}{dt} = 0, \tag{2.17}$$

$$\frac{d\tilde{\phi}_2}{dt} = \phi_1\bar{\phi}_1\cosh 2h - (\phi_1'\bar{\eta}_1 + \bar{\phi}_1'\eta_1)\sinh h. \tag{2.18}$$

To obtain an approximate solution to (2.12)–(2.16) we use the method of multiple time scales (Nayfeh 1965, 1968). Thus we assume that

$$\eta_1(t) = \eta_{10}(T_0, T_2) + \epsilon^2\eta_{12}(T_0, T_2) + \dots, \tag{2.19}$$

$$\phi_1(t) = \phi_{10}(T_0, T_2) + \epsilon^2\phi_{12}(T_0, T_2) + \dots, \tag{2.20}$$

where $T_0 = t$ and $T_2 = \epsilon^2 t$. We assume similar expansions for $\eta_2(t)$ and $\phi_2(t)$. Substitution of (2.19) and (2.20) in (2.13) and (2.14) leads to equations for η_{10} and ϕ_{10} whose solutions contain arbitrary functions of the time T_2. These arbitrary functions are determined by requiring that $\epsilon^2\eta_{12}$ and $\epsilon^2\phi_{12}$ be small corrections to η_{10} and ϕ_{10} respectively for all T_0. In other words, η_{12}/η_{10} and ϕ_{12}/ϕ_{10} are bounded for all T_0. Since we shall be interested in second-order uniformly valid solutions, we need only to inspect the equations for η_{12} and ϕ_{12} without solving for them in order to determine the arbitrary functions in η_{10} and ϕ_{10}. In the next two sections, we will determine expansions for travelling and standing waves, respectively.

3. Expansion for travelling waves

In this section we consider travelling waves; i.e. η_1, ϕ_1, η_2, and ϕ_2 are complex functions. Substituting (2.19) and (2.20) in (2.13) and (2.14) leads to

$$\frac{\partial\eta_{10}}{\partial T_0} + \phi_{10}\sinh h = 0, \tag{3.1}$$

$$\frac{\partial\phi_{10}}{\partial T_0}\cosh h - (k^2-1)\,\eta_{10} = 0, \tag{3.2}$$

$$\frac{\partial \eta_{12}}{\partial T_0} + \phi_{12}\sinh h = f(T_0, T_2) - \frac{\partial \eta_{10}}{\partial T_2}, \tag{3.3}$$

$$\frac{\partial \phi_{12}}{\partial T_0}\cosh h - (k^2-1)\,\eta_{12} = g(T_0, T_2) - \frac{\partial \phi_{10}}{\partial T_2}\cosh h. \tag{3.4}$$

The equations for η_{20} and ϕ_{20} are

$$\frac{\partial \eta_{20}}{\partial T_0} + 2\phi_{20}\sinh 2h = -2\eta_{10}\phi_{10}\cosh h, \tag{3.5}$$

$$\frac{\partial \phi_{20}}{\partial T_0}\cosh 2h - (4k^2-1)\,\eta_{20} = -\tfrac{1}{2}\phi_{10}^2 - \eta_{10}\phi_{10}'\sinh h. \tag{3.6}$$

The general solutions of (3.1) and (3.2) are

$$\eta_{10} = A(T_2)\,e^{i\sigma_0 T_0}, \tag{3.7}$$

$$\phi_{10} = -i\sigma_0 A(T_2)\,e^{i\sigma_0 T_0}/\sinh h, \tag{3.8}$$

where

$$\sigma_0^2 = (k^2-1)\tanh h. \tag{3.9}$$

Substituting (3.7) and (3.8) in (3.5) and (3.6) leads to

$$\frac{\partial \eta_{20}}{\partial T_0} + 2\phi_{20}\sinh 2h = 2i\sigma_0 A^2\coth h\, e^{2i\sigma_0 T_0}, \tag{3.10}$$

$$\frac{\partial \phi_{20}}{\partial T_0}\cosh 2h - (4k^2-1)\,\eta_{20} = \frac{\sigma_0^2 A^2 e^{2i\sigma_0 T_0}}{2\sinh^2 h} - \sigma_0^2 A^2 e^{2i\sigma_0 T_0}. \tag{3.11}$$

The solutions of (3.10) and (3.11) are

$$\eta_{20} = B(T_2)\,e^{i\mu T_0} + \frac{\sigma_0^2}{\mu^2 - 4\sigma_0^2}A^2 S\, e^{2i\sigma_0 T_0}, \tag{3.12}$$

$$\phi_{20} = -\frac{i\mu}{2\sinh 2h}B(T_2)\,e^{i\mu T_0} + \frac{i\sigma_0}{\sinh 2h}\left[\coth h - \frac{\sigma_0^2}{\mu^2-4\sigma_0^2}S\right]A^2 e^{2i\sigma_0 T_0}, \tag{3.13}$$

where

$$\mu^2 = 2(4k^2-1)\tanh 2h, \tag{3.14a}$$

$$S = -4\coth h - \frac{\tanh 2h}{\sinh^2 h} + 2\tanh 2h. \tag{3.14b}$$

Elimination of ϕ_{12} from (3.3) and (3.4) yields

$$\frac{\partial^2 \eta}{\partial T_0^2} + \sigma_0^2\eta_{12} = \frac{\partial f}{\partial T_0} - g\tanh h - 2\frac{\partial^2 \eta_{10}}{\partial T_0 \partial T_2}. \tag{3.15}$$

Using (2.13*b*), (2.14*b*), (3.7), (3.8), (3.12)–(3.14), we get from (3.15)

$$\frac{\partial^2 \eta}{\partial T_0^2} + \sigma_0^2\eta_{12} = (-2i\sigma_0 A' - 8\sigma_0\sigma_2 A^2\bar{A})\,e^{i\sigma_0 T_0}, \tag{3.16}$$

where

$$\sigma_2 = \frac{\sigma_0}{8}\left\{\frac{S\sigma_0^2}{\mu^2-4\sigma_0^2}\left[\coth h + 2\coth 2h + \tanh h\left(\frac{2\cosh 3h}{\sinh h\sinh 2h} - 5\right)\right]\right.$$
$$\left. - \left(2\coth 2h\coth h + \frac{2\cosh 3h}{\sinh h \sinh 2h}\right) + 8 - \tfrac{3}{2}k^2\tanh h/\sigma_0^2\right\}. \tag{3.17}$$

The ratio of η_{12} to η_{10} is unbounded as $T_0 \to \infty$ because the particular solution of (3.16) is proportional to $2\sigma_0(iA' + 4\sigma_2 A^2\bar{A})\, T_0 \exp(i\sigma_0 T_0)$. In order that η_{12}/η_{10} be bounded for all T_0, we require that

$$iA' + 4\sigma_2 A^2 \bar{A} = 0. \tag{3.18}$$

Letting $A(T_2) = \frac{1}{2}C(T_2)\,[\exp i\theta(T_2)]$ in (3.18) leads to

$$C(T_2) = \text{constant}, \tag{3.19}$$

$$\theta(T_2) = C^2\sigma_2 T_2 + \theta_0, \tag{3.20}$$

where θ_0 is an arbitrary constant.

Using the initial conditions (2.8) and (2.9) we find that

$$\eta(x,t) = \epsilon \cos(x+\sigma t) + \tfrac{1}{2}\epsilon^2 \frac{\sigma_0^2 S}{\mu^2 - 4\sigma_0^2}\,[\cos 2(x+\sigma t) - \cos(2x+\mu t)] + O(\epsilon^3 t), \tag{3.21}$$

where
$$\sigma = \sigma_0 + \epsilon^2\sigma_2 + O(\epsilon^3). \tag{3.22}$$

In (3.21) we assumed that $B(T_2)$ is a constant within the order of error indicated. It is clear that, for positive σ_0^2, (3.21) represents travelling oscillatory waves with wave speeds of σ and $\frac{1}{2}\mu$. On the other hand, for negative σ_0^2, travelling waves are not possible, and disturbances are unstable as will be shown in the next section. Expansion (3.21) is not valid when $\sigma_0 = O(\epsilon)$ because $\epsilon^2\sigma_2$ and σ_0 are of the same order, and $\sigma \to \infty$ as $\sigma_0 \to 0$. Thus, we cannot determine the cut-off wave-number from (3.22). We will determine the cut-off wave-number, and an expansion valid when $\sigma_0 = O(\epsilon)$ in §§ 5 and 6, respectively.

If the layer thickness is infinite or $k' \to \infty$; i.e. $h \to \infty$, then

$$\eta(x,t) \to \epsilon\cos(x+\sigma t) - \frac{\epsilon^2}{2}\frac{k^2-1}{2k^2+1}\,[\cos 2(x+\sigma t) - \cos(2x+\mu t)] + O(\epsilon^3 t), \tag{3.23}$$

$$\sigma = (k^2-1)^{\frac{1}{2}}\left\{1 + \frac{\epsilon^2}{8}\left[-2\frac{k^2-1}{2k^2+1} - \frac{3}{2}\frac{k^2}{k^2-1} + 2\right]\right\} + O(\epsilon^3). \tag{3.24}$$

Moreover, if $k \to \infty$ (i.e. capillary waves), then

$$\eta(x,t) \to \epsilon\cos(x+\sigma t) - \frac{\epsilon^2}{8}\,[\cos 2(x+\sigma t) - \cos(2x + 2^{\frac{3}{2}}kt)] + O(\epsilon^3 t), \tag{3.25}$$

$$\sigma = k\left[1 - \frac{\epsilon^2}{16}\right]. \tag{3.26}$$

4. Expansion for standing waves

In this section, we consider standing waves; i.e. η_1, η_2, ϕ_1 and ϕ_2 are real. The general solutions for η_{10} and ϕ_{10} are

$$\eta_{10} = A(T_2)\,e^{i\sigma_0 T_0} + \bar{A}(T_2)\,e^{-i\sigma_0 T_0}, \tag{4.1}$$

$$\phi_{10} = -\frac{i\sigma_0}{\sinh h}\,[A\,e^{i\sigma_0 T_0} - \bar{A}\,e^{-i\sigma_0 T_0}]. \tag{4.2}$$

where $\sigma_0^2 = (k^2-1)\tanh h$. Using (4.1) and (4.2), we find that the solutions of (3.5) and (3.6) are

$$\left.\begin{aligned}\eta_{20} = {} & \frac{\sigma_0^2 S}{\mu^2-4\sigma_0^2}\,(A^2\,e^{2i\sigma_0 T_0}+\bar{A}^2\,e^{-2i\sigma_0 T_0}) \\ & +\frac{2\sigma_0^2}{\mu^2}\tanh 2h\left(2+\frac{1}{\sinh^2 h}\right)A\bar{A}+B(T_2)\,e^{i\mu T_0}+\bar{B}(T_2)\,e^{-i\mu T_0},\end{aligned}\right\} \tag{4.3}$$

$$\left.\begin{aligned}\phi_{20} = {} & \frac{i\sigma_0}{\sinh 2h}\left[\coth h-\frac{\sigma_0^2 S}{\mu^2-4\sigma_0^2}\right](A^2\,e^{2i\sigma_0 T_0}-\bar{A}^2\,e^{-2i\sigma_0 T_0}) \\ & -\frac{i\mu}{2\sinh 2h}\,(B\,e^{i\mu T_0}-\bar{B}\,e^{-i\mu T_0}).\end{aligned}\right\} \tag{4.4}$$

Substituting (4.1)–(4.4) in (3.15) leads to

$$\left.\begin{aligned}\frac{\partial^2\eta_{12}}{\partial T_0^2}+\sigma_0^2\eta_{12} = {} & (-2i\sigma_0 A'-32\sigma_0\tilde{\sigma}_2 A^2\bar{A})\,e^{i\sigma_0 T_0}+(2i\sigma_0\bar{A}'-32\sigma_0\tilde{\sigma}_2\bar{A}^2 A)\,e^{-i\sigma_0 T_0} \\ & +\text{terms proportional to } [\exp(\pm 3i\sigma_0 T_0),\quad \exp(\pm i\sigma_0 T_0\pm i\mu T_0)],\end{aligned}\right\} \tag{4.5}$$

$$\left.\begin{aligned}\tilde{\sigma}_2 = {} & \frac{\sigma_0}{8}\left\{\frac{1}{2}\frac{\sigma_0^2 S}{\mu^2-4\sigma_0^2}\left[\tfrac{1}{2}\coth h+\coth 2h+\tanh h\left(\frac{\cosh 3h}{\sinh h\sinh 2h}-2\tfrac{1}{2}\right)\right]\right. \\ & -\frac{1}{2}\frac{\sigma_0^2}{\mu^2}\left(2+\frac{1}{\sinh^2 h}\right)(\coth h+\tanh h)\tanh 2h-\tfrac{9}{8}k^2\tanh h/\sigma_0^2 \\ & \left.-\tfrac{1}{2}\coth h\coth 2h-\frac{1}{2}\frac{\cosh 3h}{\sinh h\sinh 2h}+1\right\}.\end{aligned}\right| \tag{4.6}$$

In order that η_{12}/η_{10} be bounded for all T_0, we require the vanishing of the coefficients of $e^{\pm i\sigma_0 T_0}$ on the right-hand side of (4.5). Thus,

$$iA'+16\tilde{\sigma}_2 A^2\bar{A} = 0. \tag{4.7}$$

Letting $A(T_2) = \frac{1}{2}C(T_2)\exp i\theta(T_2)$, where C and θ are real functions, we get

$$C(T_2) = \text{constant}, \tag{4.8}$$

$$\theta(T_2) = 4C^2\tilde{\sigma}_2 T_2+\theta_0, \tag{4.9}$$

where θ_0 is an arbitrary constant.

Using the initial conditions (2.8) and (2.9), we get

$$\eta(x,t) = \epsilon\cos\sigma t\cos x+\epsilon^2 P(t)\cos 2x+O(\epsilon^3 t), \tag{4.10a}$$

$$P(t) = \frac{1}{4}\frac{\sigma_0^2 S}{\mu^2-4\sigma_0^2}(\cos 2\sigma t-\cos\mu t)+\frac{1}{4}\frac{\sigma_0^2}{\mu^2}\left(2+\frac{1}{\sinh^2 h}\right)(1-\cos\mu t) \tag{4.10b}$$

and

$$\sigma = \sigma_0+\epsilon^2\tilde{\sigma}_2+O(\epsilon^3). \tag{4.11}$$

The function $B(T_2)$ is assumed to be a constant in (4.10) within the order of error indicated. If σ_0^2 is positive, and not near zero, (4.10) is valid for times as large as $O(\epsilon^{-2})$, and represents oscillatory standing waves with frequencies σ and μ. On the other hand, if σ_0^2 is negative, (4.10) represents growing waves, and it is valid only for short times. As time increases, the second-order term becomes of the same order and then dominates the first-order term contrary to our assumptions about the orders of magnitude when we carried out the expansions. Moreover, (4.11) is not valid when $k-1 = O(\epsilon^2)$ because, as $k \to 1$, $\sigma \to \infty$. We will obtain an expansion valid when $k-1 = O(\epsilon^2)$ in § 6.

Limiting cases

For an infinite layer thickness or $k' \to \infty$, i.e. $h \to \infty$,

$$\eta(x,t) = \epsilon \cos \sigma t \cos x + \epsilon^2 \left[-\frac{1}{4}\frac{k^2-1}{2k^2+1}(\cos 2\sigma t - \cos \mu t) + \frac{1}{4}\frac{k^2-1}{4k^2-1}(1-\cos \mu t) \right] \times \cos 2x + O(\epsilon^3 t), \quad (4.12)$$

$$\sigma = \sigma_0 \left(1 - \frac{\epsilon^2}{16}\beta\right) + O(\epsilon^3), \quad (4.13)$$

where $\sigma_0^2 = k^2 - 1$, $\mu^2 = 2(4k^2-1)$, and

$$\beta = \frac{9}{4}\frac{k^2}{k^2-1} + \frac{3k^2}{2k^2+1} + 2\frac{k^2-1}{4k^2-1}. \quad (4.13a)$$

Equations (4.12) and (4.13) agree with those of Rajappa (1967).

Squaring (4.13) yields

$$\sigma^2 = \sigma_0^2 \left(1 - \frac{\epsilon^2}{8}\beta\right) + O(\epsilon^3). \quad (4.14)$$

Although $\sigma \to \infty$, σ^2 tends to a finite value. Therefore, Rajappa assumed (4.12) and (4.14) to be valid for all k. Letting $\sigma^2 = 0$, he obtained the following equation for the cut-off wave-number:

$$k^2 = 1 + \tfrac{9}{32}\epsilon^2 + O(\epsilon^3). \quad (4.15)$$

However, although σ^2 is finite as $k \to 1$, it is still not valid when $k^2 - 1 = O(\epsilon^2)$ because the order-of-magnitude assumptions used in carrying out the expansion do not hold. Not only is the ratio of the second term to the first term not much smaller than one, but it may be much larger than one. Moreover, expanding (4.12) and (4.13) for small ϵt at $k = 1$ gives

$$\eta = \epsilon(1 + \tfrac{9}{64}\epsilon^2 t^2)\cos x + \tfrac{3}{64}\epsilon^3(\cos 2 . 6^{\frac{1}{2}} t - 1)\cos 3x, \quad (4.16)$$

whereas the straightforward expansion at $k = 1$ yields

$$\eta = \epsilon(1 + \tfrac{3}{16}\epsilon^2 t^2)\cos x + \tfrac{3}{64}\epsilon^3(\cos 2 . 6^{\frac{1}{2}} t - 1)\cos 3x. \quad (4.17)$$

Hence the expansions (4.12) and (4.13) are not valid when $k^2 - 1 = O(\epsilon^2)$, and consequently the cut-off wave-number given by (4.14) is incorrect.

If we expand the $\cos \sigma t$ and $\cos 2\sigma t$ in (4.12) for small ϵ, we get

$$\eta(x,t) = \epsilon \cos \sigma_0 t \cos x + \epsilon^2 \left[-\frac{1}{4}\frac{k^2-1}{2k^2+1}(\cos 2\sigma_0 t - \cos \mu t) + \frac{1}{4}\frac{k^2-1}{4k^2-1}(1-\cos \mu t) \right] \cos 2x + \frac{\epsilon^2 \sigma_0}{16}\beta t \sin \sigma_0 t \cos x. \quad (4.18)$$

The first two terms in (4.18) agree with those obtained by Emmons *et al.* (1960), and the third term here agrees with a similar term in their third-order expression.

It is clear that, for positive σ_0^2, the third term in (4.18) is of the same order as the first- and second-order terms when $\epsilon^2 t = O(1)$ and $\epsilon t = O(1)$, respectively. Therefore this expansion is valid only for short times, and breaks down for times as large as $O(\epsilon^{-1})$; consequently it cannot be used to explain the over-stability behaviour observed by Emmons *et al.* (1960).

If $k \to \infty$, we get the capillary waves

$$\eta(x,t) = \epsilon \cos \sigma t + \frac{\epsilon^2}{16}(1 + \cos \mu t - 2\cos 2\sigma t)\cos 2x + O(\epsilon^3 t), \tag{4.19}$$

where

$$\mu^2 = 8k^2, \quad \sigma = k(1 - \tfrac{17}{64}\epsilon^2). \tag{4.20}$$

5. Stationary solution, cut-off wave-number

For stationary solutions, (2.4) and (2.5) become

$$\phi_y = \phi_x \eta_x, \tag{5.1}$$

$$\eta + k^2\eta_{xx}(1+\eta_x^2)^{-\frac{3}{2}} = \tfrac{1}{2}(\phi_x^2 + \phi_y^2) \tag{5.2}$$

on $y = \eta(x)$. We let

$$\eta(x) = \epsilon \cos x + \epsilon^3 \eta_3 + \epsilon^5 \eta_5 + \ldots, \tag{5.3}$$

$$\phi(x,y) = \epsilon\phi_0(x,y) + \epsilon^3\phi_1(x,y) + \ldots, \tag{5.4}$$

$$k^2 = 1 + \epsilon^2\alpha_2 + \epsilon^4\alpha_4 + \ldots. \tag{5.5}$$

Substituting (5.3)–(5.5) in (2.2), (5.1) and (5.2), we find that

$$\phi_n(x,y) \equiv 0 \quad \text{for} \quad n = 0, 1, 2, \ldots. \tag{5.6}$$

The equation for η_3 is

$$\eta_3'' + \eta_3 = (\alpha_2 - \tfrac{3}{8})\cos x + \tfrac{3}{8}\cos 3x. \tag{5.7}$$

In order that η_3/η_1 be bounded for all x, and hence yield a uniformly valid expansion for all x, we require the vanishing of the secular producing term in (5.7). Hence

$$\alpha_2 = \tfrac{3}{8}. \tag{5.8}$$

The solution of η_3 is

$$\eta_3 = -\tfrac{3}{64}\cos 3x. \tag{5.9}$$

With (5.9) the equation for η_5 becomes

$$\eta_5'' + \eta_5 = (\alpha_4 + \tfrac{21}{512})\cos x + \text{higher harmonics}. \tag{5.10}$$

In order that η_5/η_1 be bounded for all x, we require that

$$\alpha_4 = -\tfrac{21}{512}. \tag{5.11}$$

Summarizing, the cut-off wave-number is given by

$$k^2 = 1 + \tfrac{3}{8}\epsilon^2 - \tfrac{21}{512}\epsilon^4 + O(\epsilon^6); \tag{5.12}$$

or in dimensional quantities:

$$k'^2 = k_c^2[1 + \tfrac{3}{8}a^2k_c^2 + \tfrac{51}{512}a^4k_c^4] + O(a^6). \tag{5.13}$$

This cut-off wave-number is different from the one obtained by Rajappa (1967).

6. Expansion valid near $k = 1$

As discussed in §§ 3 and 4, expansions (3.21) and (3.22) for travelling waves, and (4.10) and (4.11) for standing waves are valid for times as large as (ϵ^{-2}) if σ_0^2 is positive and not near zero. As $\sigma_0 \to 0$ $(k \to 1)$, σ and $\tilde{\sigma} \to \infty$. When $k-1 = O(\epsilon^2)$, $\epsilon^2\sigma_2$ in (3.22) and $\epsilon^2\tilde{\sigma}_2$ in (4.11) are of the same order as σ_0 contrary to our implicit assumption that $\epsilon^2\sigma_2$ and $\epsilon^2\tilde{\sigma}_2$ are small corrections to σ_0.

To determine an expansion valid when $k-1 = O(\epsilon^2)$, we notice that, although $T_0 = t = O(1)$, $\sigma_0 T_0 = \sigma_0 t = O(\epsilon)$. Hence the appropriate time scales in this case are $T_1 = \epsilon t$ and $T_2 = \epsilon^2 t$. We find here the variation of $\eta(x,t)$ and $\phi(x,y,t)$ with respect to the time scale T_1 which we denote by τ. We let

$$k^2 - 1 = \alpha\epsilon^2, \tag{6.1}$$

where $\alpha = O(1)$ and may be positive or negative.

Equations (4.1)–(4.4) indicate that $\eta_{10} = O(1)$, $\phi_{10} = O(\sigma_0)$, $\eta_{20} = O(\sigma_0^2)$ and $\phi_{20} = O(\sigma_0)$. Therefore we assume the following expansions:

$$\eta_1(t) = \eta_{10}(\tau) + \epsilon^2\eta_{12}(\tau) + \ldots, \tag{6.2}$$

$$\phi_1(t) = \epsilon\phi_{10}(\tau) + \ldots, \tag{6.3}$$

$$\eta_2(t) = \epsilon^2\eta_{20}(\tau) + \ldots, \tag{6.4}$$

$$\phi_2(t) = \epsilon\phi_{20}(\tau) + \ldots. \tag{6.5}$$

Substituting these expansions in (2.12)–(2.16) leads to the following equations for η_{10} and ϕ_{10}:

$$\frac{d\eta_{10}}{d\tau} + \phi_{10}\sinh h = 0, \tag{6.6}$$

$$\frac{d\phi_{10}}{d\tau}\cosh h - \alpha\eta_{10} = -\tfrac{3}{2}\eta_{10}^2\bar{\eta}_{10}. \tag{6.7}$$

Eliminating ϕ_{10} from (6.6) and (6.7) leads to

$$\frac{d^2\eta_{10}}{d\tau^2} + (\alpha\eta_{10} - \tfrac{3}{2}\eta_{10}^2\bar{\eta}_{10})\tanh h = 0. \tag{6.8}$$

For travelling waves, we let $\eta_{10} = \tfrac{1}{2}z(\tau)\,e^{i\theta(\tau)}$, separate real and imaginary parts in (6.8), and obtain

$$d\theta/d\tau = 0, \tag{6.9}$$

$$\frac{d^2 z}{d\tau^2} + (\alpha z - \tfrac{3}{8}z^3)\tanh h = 0. \tag{6.10}$$

Equation (6.9) shows that the travelling waves are reduced to standing waves because θ is a constant. Therefore, analysis of (6.10) yields solutions for travelling as well as standing waves. From (2.8) and (2.9), we get

$$z(0) = 1 \quad \text{and} \quad dz(0)/d\tau = 0. \tag{6.11}$$

A first integral of (6.10) using (6.11) is

$$\left(\frac{dz}{d\tau}\right)^2 = \tfrac{3}{16}\tanh h(z^2-1)(z^2+1-16\alpha/3). \tag{6.12}$$

Since $z(\tau)$ is real and $z(0) = 1$, z cannot decrease from 1 if $(16\alpha/3-1) < 1$, otherwise $dz/d\tau$ will be imaginary. As a result $dz/d\tau$ is positive, and hence z increases without limit. In this case, we have instability, and the solution of (6.11) and (6.12) is (Pierce & Foster 1957, pp. 73–4)

$$\mathrm{sn}^{-1}(1/z, b_1) = \mathrm{sn}^{-1}(1, b_1) - \tfrac{1}{4}(3\tanh h)^{\frac{1}{2}}\tau, \tag{6.13}$$

for $b_1^2 = 16\alpha/3-1, b_1 < 1$. In the case of $b_2^2 = 1-16\alpha/3$, the solution of (6.11) and (6.12) becomes

$$\mathrm{cn}^{-1}[1/z, b_2(1+b_2^2)^{-1}] = \tfrac{1}{4}[3(1+b_2^2)\tanh h]^{\frac{1}{2}}\tau. \tag{6.14}$$

On the other hand, if $16\alpha/3-1 = b_3^2 > 1$, then z^2 is bounded between 0 and 1. In this case, we have stability, and the solution of (6.11) and (6.12) is

$$\mathrm{sn}^{-1}(z, 1/b_3) = \mathrm{sn}^{-1}(1, 1/b_3) - \tfrac{1}{4}(3\tanh h)^{\frac{1}{2}} b_3\tau. \tag{6.15}$$

Therefore

$$\alpha = \tfrac{3}{8} \tag{6.16}$$

separates the stable and unstable regions. Using (6.1), we find that the cut-off wave-number is given by

$$k^2 = 1+\tfrac{3}{8}\epsilon^2, \tag{6.17}$$

which is in agreement within $O(\epsilon^2)$ with the cut-off wave-number we obtained earlier (equation (5.12)).

7. Discussion and conclusions

The present theory shows that the cut-off wave-number is amplitude dependent, contrary to the predictions of the linear theory of Bellman & Pennington (1954). The cut-off wave-number is given by

$$k_c' = k_c[1+\tfrac{3}{8}a^2k_c^2+\tfrac{51}{512}a^4k_c^4]^{\frac{1}{2}}+O(a^6), \tag{7.1}$$

where k_c is the linear cut-off wave-number and a is the disturbance amplitude. Therefore disturbances with wave-numbers equal to k_c still grow.

In non-dimensional quantities, and for $k-1 = O(\epsilon^\gamma)$, $\gamma < 2$, travelling waves are given by

$$\eta(x,t) = \epsilon\cos[(\sigma_0+\epsilon^2\sigma_2)t+x]+O(\epsilon^2), \tag{7.2}$$

whereas standing waves are given by

$$\eta(x,t) = \epsilon\cos(\sigma_0+\epsilon^2\tilde{\sigma}_2)t\cos x+O(\epsilon^2). \tag{7.3}$$

For $k > k_c'/k_c$ but $k-1 = O(\epsilon^2)$, the wave velocity of a travelling wave is so slow that it appears as a standing wave. The solution for η in this case is

$$\eta(x,t) = \epsilon z(t;\,\epsilon)\cos x+O(\epsilon^2), \tag{7.4}$$

where $z(t;\,\epsilon)$ is periodic and given by an elliptic integral.

Therefore travelling as well as standing disturbances with wave-numbers larger that k_c' oscillate with time-independent amplitudes and hence are stable. However, the wave velocity in the case of travelling waves and the frequency in the case of standing waves are amplitude dependent. For large h and k, $\sigma_2 = -\frac{1}{16}k$ and $\tilde{\sigma}_2 = -\frac{17}{64}k$. As k decreases the difference between σ_2 and $\tilde{\sigma}_2$ decreases, and vanishes when $k-1 = O(\epsilon^2)$. Hence, the non-linear standing waves (7.3) cannot be obtained by the superposition of the travelling waves (7.2) as in the linear case.

Equation (7.1) shows that the cut-off wave-number is independent of the layer thickness. However, decreasing the layer thickness decreases the growth rate of unstable disturbances, and hence is stabilizing.

The physical model used in the present theory, as in the theories of Emmons *et al.* (1960) and Rajappa (1967), does not explain the overstability behaviour observed by Emmons *et al.* (1960). They observed that disturbances with $k > 1$ oscillate in time with an ever-increasing amplitude; whereas the theory indicates oscillations with time-independent amplitudes.

To explain the overstability behaviour, let us first describe briefly the experimental apparatus of Emmons *et al.* (1960). They used an apparatus consisting of an aluminium frame with glass front and back walls. The frame is partially filled with liquid, and restrained to move in a vertical direction between two guide columns. They accelerated this frame and its contents by holding the frame at the top of the two guides by a steel wire while tension was applied to rubber tubes attached to the bottom of the frame. They began the runs by passing an electric current through the steel wire, melting it and releasing the frame. Since the force applied by the rubber tubes is proportional to their extensions, the acceleration applied to the frame and its contents is

$$\tilde{g} = z_0 \omega'^2 \cos \omega' t' - g_0, \tag{7.5}$$

where g_0 is the gravitational acceleration and z_0 and ω'^2 are the initial extension and the spring constant of the rubber tubes. Equation (7.5) shows that the acceleration applied to the system by the rubber tubes is not constant, and hence we take its time variation into account in the following analysis.

If we assume that

$$g' = z_0 \omega'^2 - g_0, \quad q' = z_0 \omega'^2/g', \quad e = g_0/g', \quad \omega = \omega'(g'k')^{-\frac{1}{2}}, \tag{7.6}$$

then (2.5) is modified to

$$-(q' \cos \omega t - e)\,\eta - \phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) = k^2 \eta_{xx}(1 + \eta_x^2)^{-\frac{3}{2}}. \tag{7.7}$$

Then (3.2) becomes

$$\frac{\partial \phi_{10}}{\partial T_0} \cosh h - [k^2 + e - q' \cos \omega T_0]\,\eta_{10} = 0. \tag{7.8}$$

Eliminating ϕ_{10} between (3.1) and (7.8) leads to

$$\frac{\partial^2 \eta_{10}}{\partial T^2} + (\delta - 2q \cos 2T)\,\eta_{10} = 0, \tag{7.9}$$

where $T = \frac{1}{2}\omega T_0$, $\delta = 4(k^2 + e) \tanh h/\omega^2$, and $q = 2q' \tanh h/\omega^2$. This is the familiar Mathieu equation (Whittaker & Watson 1962, pp. 404–28) which has stable or unstable solutions depending on the values of δ and q, and hence the disturbance wave-number and the rubber spring constant. The unstable solutions are either oscillating with exponentially growing amplitudes, or non-oscillating exponentially increasing. Since there were small variations in the initial conditions (Emmons, Chang & Watson 1960), it is possible that one of these wave-numbers, or the primary wave-number, corresponds to an unstable solution of (7.9), and hence produced the overstable behaviour.

Another possible explanation for the overstability behaviour is the interaction among different Fourier components which might be present due to the small variations in the initial conditions. However, the limitation imposed by (2.8) on the theory does not allow the assessment of this possibility.

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